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On blow-analytic equivalence

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This is a resume for the talk, with the title above, at 29 November 2007 at RIMS workshop. This is a joint work with Laurentiu Paunescu.

Motivated by the classification problem of analytic function germs, T.-C. Kuo ([31]) introduced the notions of blow-analytic maps and blow-analytic equivalence. We start the article explaining this motivation to define blow-analytic equivalence.

He discovered a finite classification theorem for analytic function germs with isolated singularities and also shows some important triviality theorems. We are going to report several facts known now about the blow-analytic triviality and invariants.

We then discuss Lipschitz property of blow-analytic maps and show blow-analytic homeomorphism can be far from Lipschitz map. We also discuss exotic pathologies on a blow-analytic homeomorphism: this is illustrated by the examples in §7. We then introduce a strengthened notion, called blow-analytic isomorphism, and discuss the behavior of their jacobians.

In §8, we present a version of the Inverse Mapping Theorem for blow-analytic isomorphisms.

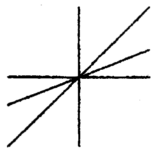
1. Motivations

The notion of blow-analytic equivalence arises from attempts to classify analytic function germs. One is tempted to use the following equivalence relation.

Definition 1.1. Let $k = 0, 1, 2, \dots, \infty, \omega$. We say that two analytic function-germs $f, g : \mathbf{R}^n, 0 \to \mathbf{R}, 0$ are C^k*-equivalent* if there is a C^k-diffeomorphism-germ $h : \mathbf{R}^n, 0 \to \mathbf{R}^n, 0$ so that $f = g \circ h$.

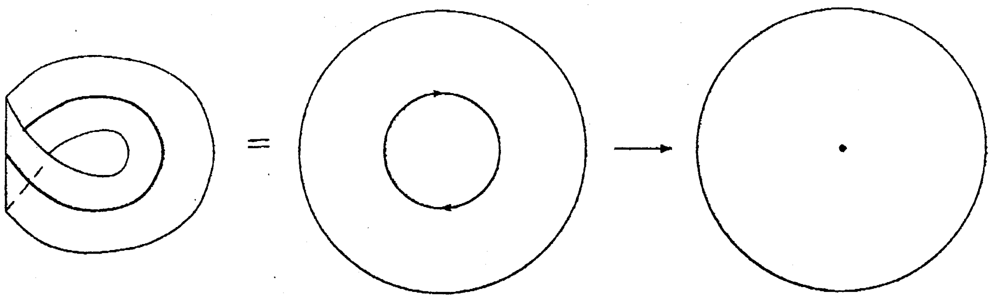
However, the following example, due to H. Whitney, shows that the C^1-equivalence is already too fine for the classification purpose.

Example 1.2 ([41]). Consider the functions $f_t : \mathbf{R}^2, 0 \to \mathbf{R}$, $0 < t < 1$, defined by $f_t(x, y) = xy(y - x)(y - tx)$. Then f_t is C^1-equivalent to $f_{t'}$, if and only if $t = t'$.



As for the C^0-equivalence, the functions $(x, y) \mapsto x^2 + y^{2k+1}$, $k \geq 1$, for instance, are C^0-equivalent to the regular function $(x, y) \mapsto y$. Hence it seems hopeless to expect a decent classification theory.

Now we consider the blowing-up $\pi : M \to \mathbf{R}^2$ at 0. This map is illustrated by the following picture.



The anti-podal points of the inner circle of the annulus in the middle figure are identified to obtain the Möbius strip in the left figure. Collapsing the inner circle to a point, yields a mapping from the Möbius strip to the disk at the right. This is called the blowing-up of the disk at its centre point. One can introduce local coordinates on the Möbius strip and then the above mapping can be expressed as a real analytic map, as follows. Let $M = \{(x, y) \times [\xi : \eta] \in D^2 \times P^1 : x\eta = y\xi\}$, where D^2 is a 2-dimensional disk and P^1 is the real projective line. The restriction of the projection $(x, y) \times [\xi : \eta] \mapsto (x, y)$ to M is the desired π. For the functions f_t in Example 1.2, all $f_t \circ \pi$ are C^ω- equivalent to each other ([31]).

2. Definition of blow-analytic map

2.1. A naive introduction.

Definition 2.1 (Blowing-up). Let U be a disk in $\mathbf{R}^n$ with analytic coordinates $x_1, \ldots, x_n$, and let $C \subset U$ be the locus $x_1 = \cdots = x_k = 0$. Let $[\xi_1 : \cdots : \xi_k]$ be homogeneous coordinates of the real projective space P^{k-1} and let $\widetilde{U} \subset U \times P^{k-1}$ be the nonsingular manifold defined by

$$\widetilde{U} = \{(x_1, \ldots, x_n) \times [\xi_1, \ldots, \xi_k] : x_i\xi_j = x_j\xi_i,\ 1 \leq i, j \leq k\}.$$

The projection $\pi : \widetilde{U} \to U$ on the first factor is clearly an isomorphism away from C. The manifold $\widetilde{U}$, together with the map $\pi : \widetilde{U} \to U$ is called the *blowing-up* with nonsingular center C. It is well-known that the blowing-up $\pi : \widetilde{U} \to U$ is independent of the coordinates chosen in U. This allows us to globalize the definition. Let M be a real analytic manifold of dimension n and C a submanifold of codimension k. Let $\{U_\alpha\}$ be a collection of disks in M covering C such that in each disc U_α the submanifold $C \cap U_\alpha$ may be given as the locus $(x_1 = \cdots = x_k = 0)$, and let $\pi_\alpha : \widetilde{U}_\alpha \to U_\alpha$ be the blowing-up with center $C \cap U_\alpha$. We then have isomorphisms

$$\pi_{\alpha\beta} : \pi_\alpha^{-1}(U_\alpha \cap U_\beta) \to \pi_\beta^{-1}(U_\alpha \cap U_\beta),$$

and we can patch together $\widetilde{U}_\alpha$ to form a manifold $\widetilde{U} = \bigcup_{\pi_{\alpha\beta}} \widetilde{U}_\alpha$ with map $\pi : \widetilde{U} \to \bigcup U_\alpha$. Since π is an isomorphism away from C, we can take $\widetilde{M} = \widetilde{U} \cup_\pi (M - C)$; $\widetilde{M}$, together with the map $\pi : \widetilde{M} \to M$ extending π on $\widetilde{U}$ and the identity on $M - C$, is called the *blowing-up* of M with center C. We call $E = \pi^{-1}(C)$ the *exceptional divisor* of the blowing-up π.

Let M be a real analytic manifold. Take a function f defined on M except possibly on some nowhere dense subset of M. We often denote this function by $f : M \dashrightarrow \mathbf{R}$ and say that f is defined almost everywhere.

Definition 2.2. Let $\pi : \widetilde{M} \to M$ be a locally finite composition of blowing-ups with nonsingular centers. We say that $f : M \dashrightarrow \mathbf{R}$ is *blow-analytic via* π if $f \circ \pi$ has an analytic extension on $\widetilde{M}$. We say that f is *blow-analytic* if there is $\pi : \widetilde{M} \to M$, a locally finite composition of blowing-ups with nonsingular centers, so that f is blow-analytic via π.

Many functions, used as counterexamples in Calculus, are blow-analytic. Some of them are as follows.

Example 2.3. (i) $f(x,y) = \dfrac{xy}{x^2+y^2}$, $(x,y) \neq (0,0)$. This function f is not continuously extendable at the origin. It is clearly blow-analytic via the blowing-up at the origin.
(ii) $f(x,y) = \dfrac{x^2y}{x^4+y^2}$, $(x,y) \neq (0,0)$. This function is not continuously extendable at the origin, although all directional derivatives exist, if we define $f(0,0) = 0$. This function f is also blow-analytic.
(iii) $f(x,y) = \dfrac{xy(x^2-y^2)}{x^2+y^2}$, $(x,y) \neq (0,0)$. This function is continuously extendable at the origin, but the second order derivatives depend on the order of differentiation:

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0).$$

This function f is also blow-analytic via the blowing-up at the origin.

Example 2.4 ([1]). Another typical example of blow-analytic function is $f(x,y) = \sqrt[3]{x^4+y^4}$. The zero set of $z^3 + (x^2+y^2)z + x^3$ is also the graph of a blow-analytic function $z = g(x,y)$.

The notion of blow-analytic map between real analytic manifolds is defined using local coordinates.

Definition 2.5. Let X, Y be real analytic manifolds. We say that $f : X \to Y$ is a *blow-analytic homeomorphism* (bah, for short) if f is a homeomorphism and that both f and f^{-1} are blow-analytic.

Definition 2.6. Let $f, g : \mathbf{R}^n, 0 \to \mathbf{R}, 0$ be analytic functions. We say that f and g are *blow-analytically equivalent* if there is a blow-analytic homeomorphism $h : \mathbf{R}^n, 0 \to \mathbf{R}^n, 0$ so that $f = g \circ h$.

Note that h preserves the zero sets of f and g. The equivalence relation determined by the above relation on the set of analytic function-germs $\mathbf{R}^n, 0 \to \mathbf{R}, 0$ will be called *the blow-analytic equivalence.*

Example 2.7. (i) Consider the map $f : \mathbf{R}^2, 0 \to \mathbf{R}^2, 0$ defined by

$$(x,y) \mapsto \frac{1}{x^2+y^2}(x^3, y^3).$$

The map f is continuously extendable at the origin and blow-analytic. The extension is a homeomorphism. But the inverse is not blow-analytic. In fact, f^{-1} is given by

$$(X,Y) \mapsto (X^{\frac{2}{3}} + Y^{\frac{2}{3}})(X,Y).$$

(ii) Consider the map $f : \mathbf{R}^2, 0 \to \mathbf{R}^2, 0$ defined by

$$(x, y) \mapsto (x^2 + y^2)(x, y).$$

The map f is analytic and a homeomorphism. But the inverse is not blow-analytic. In fact, f^{-1} is given by

$$(X, Y) \mapsto (X^2 + Y^2)^{-1/3}(X, Y).$$

Problem 2.8. Classify the analytic function-germs by blow-analytic equivalence.

2.2. **Real v.s. complex.**

Remark 2.9. Let $h : \mathbf{R}^n, 0 \to \mathbf{R}^n, 0$ be a blow-analytic homeomorphism. Let $\pi_i : M_i \to \mathbf{R}^n$, $i = 1, 2$, be compositions of blowing-ups with nonsingular centers so that $h \circ \pi_1$ and $h^{-1} \circ \pi_2$ are analytic. It is natural to expect that, by repeating blowing-ups of M_i at nonsingular centers, if necessary, there will be an analytic isomorphism H between $\tilde{M}_1$ and $\tilde{M}_2$ which induces h. In other words, we expect to have the following commutative diagram:

$$\begin{tikzcd} \tilde{M}_1 \arrow[r, "H"] \arrow[d, "\tilde{\pi}_1"'] & \tilde{M}_2 \arrow[d, "\tilde{\pi}_2"] \\ \mathbf{R}^n \arrow[r, "h"] & \mathbf{R}^n \end{tikzcd}$$

Unfortunately, it is not known whether this is true or not.

Let $\mu : M \to N$ be a proper analytic map between real analytic manifolds. It is known that there are complexifications M^* and N^* of M, N, respectively, and a holomorphic map-germ $\mu^* : M^*, M \to N^*, N$ so that $\mu^*|M = \mu$. (See [23], page 208.)

In complex analytic geometry, a holomorphic map which is bimeromorphic is often called a *modification*. Let M^*, N^* be complex analytic manifolds with anti-holomorphic involutions σ_M, σ_N. We denote the fixed point sets of σ_M, σ_N by M, N, respectively. Let $\pi^* : M^* \to N^*$ be a proper modification so that $\sigma_N \circ \pi^* = \pi^* \circ \sigma_M$. We take its real part (restriction to M) and denote it by $\pi : M \to N$. In this paper, we call such a modification a *complex modification*.

In the setup in Remark 2.9, we can take the fiber product of $h \circ \pi_1$ and π_2 (or π_1 and $h^{-1} \circ \pi_2$) and obtain the following diagram:

$$\begin{tikzcd} & M_1 \arrow[r, "\pi_1"] & \mathbf{R}^n \arrow[d, "h"] \\ M \arrow[ur] \arrow[dr] & & \\ & M_2 \arrow[r, "\pi_2"] & \mathbf{R}^n \end{tikzcd}$$

But we do not know whether M has a complexification so that the composed maps $M \to M_i \to \mathbf{R}^n$, $i = 1, 2$, are complex modifications, even though one can take proper complexifications of π_i, $i = 1, 2$. One can say that these compositions are real modifications in the following sense. We say $\mu : M \to N$ is a *real modification*, if one can take a representative of a complexification μ^* which is an isomorphism everywhere except on a nowhere dense subset of a neighbourhood of M in M^*. Clearly a complex modification is a real modification. But it is not clear whether, or not, a real modification is a complex modification, that is, isomorphic to the real part of a complex proper modification.

Example 2.10. The following map is an analytic isomorphism, hence a real modification,

$$\mathbf{R} \to \mathbf{R}, \qquad x \mapsto x + \frac{1}{2(1+x^2)}.$$

But the homeomorphism $\mathbf{R} \to \mathbf{R}$, $x \mapsto x^3$, is not a real modification.

3. Triviality theorem

Let I be an interval in $\mathbf{R}$, which contains the origin 0. Let $F : (\mathbf{R}^n, 0) \times I \to \mathbf{R}, 0$ be an analytic function-germ. We consider the family $f_t : \mathbf{R}^n, 0 \to \mathbf{R}, 0$, $t \in I$, defined by $f_t(x) = F(x,t)$.

Definition 3.1 (Blow-analytic triviality). Let $\pi : M, E \to \mathbf{R}^n, 0$ be a proper analytic modification. We say f_t, $t \in I$, is blow-analytically trivial via π if there are a t-level preserving homeomorphism $h : (\mathbf{R}^n, 0) \times I \to (\mathbf{R}^n, 0) \times I$ and a t-level preserving analytic isomorphism $H : (M, E) \times I \to (M, E) \times I$ such that the following diagram is commutative :

$$\begin{tikzcd}
(M,E) \times I \arrow[r, "\pi \times \mathrm{id}_I"] \arrow[d, "H"'] \arrow[dr] & (\mathbf{R}^n, 0) \times I \arrow[r, "F_0"] \arrow[d, "h"] \arrow[dl] & \mathbf{R}, 0 \arrow[d, equal] \\
(M,E) \times I \arrow[r, "\pi \times \mathrm{id}_I"'] & (\mathbf{R}^n, 0) \times I \arrow[r, "F"'] & \mathbf{R}, 0
\end{tikzcd}$$

$$I$$

where $F_0 : (\mathbf{R}^n, 0) \times I \to \mathbf{R}, 0$ is the map defined by $(x,t) \mapsto f_0(x)$.

In all the cases we are interested in, $\pi : M \to \mathbf{R}^n$ is the real part of a complex proper modification $\pi^* : M^* \to \mathbf{C}^n$ defined over reals.

Consider the Taylor expansion of $f_t(x) = F(x,t)$ at 0 in $\mathbf{R}^n$:

$$f_t(x) = \sum_\nu c_\nu(t) x^\nu, \qquad \text{where } x^\nu = x_1^{\nu_1} \cdots x_n^{\nu_n},\ \nu = (\nu_1, \ldots, \nu_n).$$

We set $H_j(x,t) = \sum_{\nu:|\nu|=j} c_\nu(t) x^\nu$ where $|\nu| = \nu_1 + \cdots + \nu_n$, and assume that k is the smallest number so that $H_k(x,t)$ is not identically equal to 0.

Theorem 3.2 ([30]). *If $H_k(x,t)$ has an isolated singularity in $\mathbf{R}^n$ for any $t \in I$, then f_t, $t \in I$, is blow-analytically trivial via the blowing-up at the origin.*

Let $w = (w_1, \ldots, w_n)$ be an n-tuple of positive integers. We set

$$H_j^{(w)} = \sum_{\nu:|\nu|_w = j} c_\nu(t) x^\nu \qquad \text{where} \qquad |\nu|_w = w_1\nu_1 + \cdots + w_n\nu_n,$$

and assume that k is the smallest number so that $H_k^{(w)}$ is not identically equal to 0.

Theorem 3.3 ([14]). *If $H_k^{(w)}(x,t)$ has an isolated singularity in $\mathbf{R}^n$ for any $t \in I$, then f_t, $t \in I$, is blow-analytically trivial via a toric modification.*

See §1.5 in [36], §5 in [6], [16], about toric modifications. See [37] for a generalization of this theorem.

Example 3.4 ([4]). Consider the family $f_t(x,y,z) = z^5 + tzy^6 + y^7x + x^{15}$, $t > -15^{1/7}(7/2)^{4/5}/3$. This function is a weighted homogeneous polynomial with weight $(1,2,3)$ and weighted degree 15. This family satisfies the assumption of Theorem 3.3 and hence f_t is blow-analytically trivial. An important fact is that this family is not bilipschitz trivial near $t = 0$. See S. Koike ([28]) for a proof.

It is expected that the blow-analytic equivalence should not have moduli. Indeed T.-C. Kuo proved the following: If an analytic function $f : \mathbf{R}^n, 0 \to \mathbf{R}, 0$ defines an isolated singularity, then the number of blow-analytic equivalence classes nearby f is finite. A more precise statement is the following.

Theorem 3.5 ([31]). *Let P be a subanalytic set and let $F : (\mathbf{R}^n, 0) \times P \to \mathbf{R}, 0$ be an analytic function. If the functions $f_t : \mathbf{R}^n, 0 \to \mathbf{R}, 0$ defined by $x \mapsto F(x,t)$ have an isolated singularity for all $t \in P$, then there is a subanalytic filtration*

$$P = P_0 \supset P_1 \supset \cdots \supset P_N \supset P_{N+1} = \emptyset, \qquad \dim P_i > \dim P_{i+1},$$

such that f_t and $f_{t'}$ are blow-analytically equivalent for t, t' belonging to the same connected component of $P_i - P_{i+1}$.

K. Kurdyka ([32]) introduced the notion of arc-analytic map. We recall some fundamental facts here.

Definition 3.6 (Arc-analytic map). Let X and Y be real analytic manifolds. We say that a map $f : X \to Y$ is *arc-analytic* (a.a. for short) if $f \circ \alpha$ is analytic for any analytic map $\alpha : \mathbf{R}, 0 \to X$.

Theorem 3.7 ([1]). *Let $f : U \to \mathbf{R}$ be an arc-analytic function and U be an open subset of $\mathbf{R}^n$. If there are analytic functions $G_i(x)$, $i = 0, \ldots, p$, so that*

$$G_0(x)f(x)^p + G_1(x)f(x)^{p-1} + \cdots + G_{p-1}(x)f(x) + G_p(x) \equiv 0,$$

then f is blow-analytic.

Corollary 3.8. *An arc-analytic function with semi-algebraic graph is blow-analytic.*

Example 3.9 ([1]). The function $f(x,y) = x^3 e^{x^3/(x^2+y^2)}$ is blow-analytic. But there are no non-zero analytic functions vanishing on its graph.

Definition 3.10. Let X and Y be real analytic manifolds. We say that a map $f : X \to Y$ is *locally blow-analytic* if there is a locally finite family of analytic maps $\{\psi_i : M_i \to X\}$ with the following properties:

- ψ_i are compositions of finitely many local blowing-ups with nonsingular centers,
- there are compact subsets K_i of M_i with $\bigcup_i \psi_i(K_i) = X$, and
- $f \circ \psi_i$ are analytic.

Theorem 3.11 ([1]). *An arc-analytic function $f : U \to \mathbf{R}$ with subanalytic graph is locally blow-analytic.*

See also [40] for another proof of this theorem.

Question 3.12. Is a locally blow-analytic function $f : U \to \mathbf{R}$ blow-analytic?

When $\dim U = 2$, the answer is "yes", since local blowing-ups can be glued together to yield blowing-ups.

4. Arc lifting property

A remarkable property of blowing-up is the arc lifting property.

Definition 4.1 (Arc lifting property). Let I be an open interval in $\mathbf{R}$. Let X and Y be real analytic manifolds. We say that a map $f : X \to Y$ has *the arc lifting property* (alp. for short) if for any analytic map $\alpha : I \to Y$ there is an analytic map $\widetilde{\alpha} : I \to X$ so that $f \circ \widetilde{\alpha} = \alpha$.

$$\begin{tikzcd} & X \arrow[d, "f"] \\ I \arrow[ur, dashed, "\widetilde{\alpha}"] \arrow[r, "\alpha"'] & Y \end{tikzcd}$$

The blowing-up $\pi : \widetilde{M} \to M$ with a nonsingular center has the alp.

The blowing-up with an ideal center has the alp. because it is dominated by a composition of blowing-ups with nonsingular centers.

Example 4.2. Let $f : \mathbf{R}^2, 0 \to \mathbf{R}^2, 0$ be the map-germ defined by

$$(x, y) \mapsto \left(x, \frac{y(y^2 - x^2)}{x^2 + y^2}\right)$$

This map can be extended continuously at 0. Let $\pi : M \to \mathbf{R}^2$ be the blowing-up at the origin. Consider the map

$$F : M \to M, \quad (x, y) \times [\xi : \eta] \mapsto f(x, y) \times [\xi(\xi^2 + \eta^2) : \eta(\eta^2 - \xi^2)].$$

Here we use the same notation as that at the end of §1. It is easy to see that $\pi \circ F = f \circ \pi$. Since the image of the set of regular points of F by F is M, f has the arc lifting property. Since the jacobian of f is $\frac{-x^4+4x^2y^2+y^4}{(x^2+y^2)^2}$, which is zero along $x^2 - (2 + \sqrt{5})y^2 = 0$, $(x, y) \neq 0$, the lifting is not global.

5. Blow-analytic invariants

5.1. Singular set.

Theorem 5.1 ([39]). *Let $f, g : \mathbf{R}^n, 0 \to \mathbf{R}, 0$ be two analytic function germs, and let Σ_f and Σ_g denote their singular sets. If there is a blow-analytic homeomorphism $h : \mathbf{R}^n, 0 \to \mathbf{R}^n, 0$ with $f = g \circ h$, then $h(\Sigma_f) = \Sigma_g$. (That is h preserves the singular set.)*

However, a blow-analytic equivalence of analytic functions does not, in general, preserve their singular loci, as the following example shows.

Example 5.2. Let $f_t(x, y) = x^4 + 2t^2x^2y^2 + y^4 + x^5$, $t \in \mathbf{R}$. By Theorem 3.2, this family is blow-analytically trivial. Nevertheless, the dimension $\dim_{\mathbf{R}} \mathbf{R}\{x, y\}/\langle \frac{\partial f_t}{\partial x}, \frac{\partial f_t}{\partial y} \rangle$ changes at $t = 1$.

5.2. **Numerical invariant.** Let $f : \mathbf{R}^n, 0 \to \mathbf{R}, 0$ be an analytic function and let $\alpha : \mathbf{R}, 0 \to \mathbf{R}^n, 0$ be an analytic map. If $f \circ \alpha$ is not identically zero, then there is a positive integer k so that

$$f \circ \alpha(t) = ct^k + \text{higher order terms}, \quad c \neq 0.$$

We call k the *order of f along α* and denote it by $\mathrm{ord}_\alpha(f)$. Define $\mathrm{ord}_\alpha(f) = \infty$ when $f \circ \alpha$ is identically zero. We define $A(f)$ by

$$A(f) := \{\mathrm{ord}_\alpha(f) : \ \alpha : \mathbf{R}, 0 \to \mathbf{R}^n, 0 \text{ analytic } \}.$$

Theorem 5.3. *If two analytic function germs $f, g : \mathbf{R}^n, 0 \to \mathbf{R}, 0$ are blow-analytically equivalent, then $A(f) = A(g)$.*

Remark 5.4. Let $\mathrm{mult}_0(f)$ denote the multiplicity of f at 0, i.e., the degree of the initial polynomial of f. It is easy to show that $\mathrm{mult}_0(f) = \min A(f)$. As a consequence, the multiplicity is a blow-analytic invariant of analytic function germs. So, this theorem should be compared with Zariski's multiplicity conjecture: If two holomorphic functions $f, g : \mathbf{C}^n, 0 \to \mathbf{C}, 0$ are topologically equivalent (C^0-equivalent or C^0-V-equivalent), then $\mathrm{mult}_0(f) = \mathrm{mult}_0(g)$. This is still open. It is clear that the definition of $A(f)$ makes sense for a holomorphic function f and it is interesting to ask the following question: Is $A(f)$ a topological invariant for holomorphic functions f?

Example 5.5. Let $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. Let $f : \mathbf{K}^n, 0 \to \mathbf{K}, 0$ be the analytic function defined by $f(x_1, \ldots, x_n) = x_1^{m_1} \cdots x_n^{m_n}$. Then

$$A(f) = \Big(\sum_{i \in I} m_i \mathbf{N}\Big) \cup \{\infty\}.$$

Let $f : \mathbf{K}^n, 0 \to \mathbf{K}, 0$ be an analytic function. Let $\pi : M, E \to \mathbf{K}^n, 0$, $E = \pi^{-1}(0)$, denote a real modification. e.g., a composition of finitely many blowing-ups with nonsingular centers. We assume that $f \circ \pi$ is normal crossing, that is, $f \circ \pi$ can be locally expressed as a product of powers of a number of local coordinates. Let $(f \circ \pi)_0 = \sum_{j \in J} m_j E_j$ denote the irreducible decomposition of the zero locus of $f \circ \pi$ and $\mathcal{C}$ denote the set of subsets I of J with $E_I^* \subset E$ where $E_I^* = E_I^o \cap E$, $E_I^o = \bigcap_{i \in I} E_i - \bigcup_{j \in J - I} E_j$.

The following formula is stated in [25],Theorem I.

Theorem 5.6. $A(f) = \bigcup_{I \in \mathcal{C}} A_I(f)$ *where* $A_I(f) = (\sum_{i \in I} m_i \mathbf{N}) \cup \{\infty\}$.

Let $f : \mathbf{R}^n, 0 \to \mathbf{R}, 0$ be a real analytic function. We set

$$A^{\pm}(f) = \{\mathrm{ord}_\alpha(f) : \alpha : \mathbf{R}, 0 \to \mathbf{R}^n, 0 \text{ analytic and } \pm f \circ \alpha(t) \geq 0 \text{ near } 0\}$$

The proof of Theorem 5.3 shows $A^{\pm}(f) = A^{\pm}(g)$ if f and g are blow-analytically equivalent. In a way similar to the proof of Theorem 5.6, we obtain the following

Theorem 5.7. $A^{\pm}(f) = \bigcup_{I \in \mathcal{C}^{\pm}} A_I(f)$ *where $\mathcal{C}^{\pm}$ denotes the set of $I \in \mathcal{C}$ so that E_I^* intersects with the closure of $\{y \in M : \pm f \circ \pi(y) > 0\}$.*

5.3. **Zeta functions.** Recently S. Koike and A. Parusiński ([27]) have introduced zeta functions for the blow-analytic equivalence. In their paper ([27]), they call their zeta functions the 'motivic type invariants', since their zeta functions can be derived from zeta functions whose coefficients are motives. G. Fichou ([10]) generalizes their invariants using the virtual Poincaré polynomial. Since these are very interesting invariants, we review their results in this section. See also [35] for the virtual Betti numbers.

Let C be a category whose objects are a class of subsets of the Euclidean spaces with some good properties. We consider an invariant $\beta : C \to R$, where R is a commutative ring, with the following properties.

- $\beta(X) = \beta(X - Y) + \beta(Y)$ if Y is a closed subset in X.
- $\beta(X \times Y) = \beta(X)\beta(Y)$.

When C is the category of subanalytic subsets in Euclidean spaces which have finite homologies, the $\mathbf{Z}/2\mathbf{Z}$-Euler characteristic β with compact supports has these properties.

We say a semi-algebraic set A in a compact nonsingular real algebraic manifold M is a *$\mathcal{AS}$-subset* if for any analytic map $\alpha : (-\varepsilon, \varepsilon) \to M$, $\varepsilon > 0$, with $\alpha(0, \varepsilon) \subset A$, there is a positive number ε' so that $\alpha(-\varepsilon', 0) \subset A$. See [33] for more information about $\mathcal{AS}$-subsets.

Theorem 5.8 ([10]). *Let $\mathcal{AS}$ denote the set of all semi-algebraic $\mathcal{AS}$-subsets in compact nonsingular real algebraic manifolds. There is an invariant $\beta : \mathcal{AS} \to \mathbf{Z}[u, u^{-1}]$ with the above properties which satisfies the following:*

$$\beta(X) = \sum_k (\dim H_k(X, \mathbf{Z}/2\mathbf{Z}))u^k$$

when X is compact and nonsingular. Moreover, if two $\mathcal{AS}$-sets X, Y are Nash (i.e., semi-algebraically and analytically) equivalent, then $\beta(X) = \beta(Y)$.

Notice the following: $\beta(\emptyset) = 0$, $\beta(P^n) = 1 + u + u^2 + \cdots + u^n$, $\beta(\mathbf{R}^n) = u^n$.

Example 5.9. It is not true that $\beta(X) = k\beta(Y)$ when there is an unbranched k-fold covering $X \to Y$. Consider the double covering $S^1 \to P^1$ and observe that $\beta(S^1) = \beta(P^1) = u + 1$.

We consider the space of polynomial arcs of order k:

$$L_k := \{\alpha : \mathbf{R}, 0 \to \mathbf{R}^n, 0 : \text{polynomial of degree } k\} = \mathbf{R}^{nk}.$$

Let $f : \mathbf{R}^n, 0 \to \mathbf{R}, 0$ be an analytic function. The following spaces are algebraically constructible

$$A_k(f) := \{\alpha \in L_k : \mathrm{ord}(f \circ \alpha) = k\} \quad A_k^{\pm}(f) := \{\alpha \in L_k : f \circ \alpha = \pm t^k + \cdots\}.$$

Notice that if f and g are analytically equivalent, then $A_k(f)$ (resp. $A_k^{\pm}(f)$) and $A_k(g)$ (resp. $A_k^{\pm}(g)$) are actually isomorphic as algebraic constructible sets. Define Zeta functions by the following formulas.

$$Z_f(t) := \sum_{k \geq 1} \beta(A_k(f)) \left(\frac{t}{u^n}\right)^k \qquad Z_f^{\pm}(t) := \sum_{k \geq 1} \beta(A_k^{\pm}(f)) \left(\frac{t}{u^n}\right)^k$$

where $u = -1$ when β is the $\mathbf{Z}/2\mathbf{Z}$-Euler characteristic with compact supports ([27]), or u is an indeterminate when β is the virtual Poincaré polynomial ([10]).

Let $\pi : M, E \to \mathbf{R}^n, 0$, $E = \pi^{-1}(0)$, be a proper analytic modification so that $f\circ\pi$, $\det(d\pi)$ are in normal crossing and that π is an isomorphism over $\mathbf{R}^n - f^{-1}(0)$. We assume that $\pi^{-1}(0)$ is a normal crossing divisor. We use the notation defined in the paragraph after Example 5.5. We consider the irreducible decompositions of the zero loci of $f\circ\pi$ and $\det(d\pi)$, the jacobian determinant of π:

$$(f\circ\pi)_0 = \sum_{j\in J} m_j E_j, \qquad (\det(d\pi))_0 = \sum_{j\in J}(\nu_j - 1)E_j.$$

The following formula is often called the Denef-Loeser formula.

Theorem 5.10 ([27], [10]). *Setting $\phi(\lambda) = \lambda/(1-\lambda) = \lambda + \lambda^2 + \lambda^3 + \cdots$, we have*

$$Z_f(t) = \sum_{I\neq\emptyset} \beta(E_I^*)(u-1)^{|I|}\prod_{i\in I}\phi\Big(\frac{t^{m_i}}{u^{\nu_i}}\Big).$$

Remark 5.11. When β is the virtual Poincaré polynomial we need to assume that f is a polynomial and that π is algebraic (since we do not know that E_I^* is semi-algebraic).

It is also possible to obtain a formula for $Z_f^{\pm}(t)$ similar to Theorem 5.10. To do this, we introduce somenotation. We define $A_k^{\pm}(f, E_I^*)$ by

$$A_k^{\pm}(f, E_I^*) := p_k(\pi_*^{-1}(\mathcal{A}_k^{\pm}(f)) \cap \mathcal{L}(M, E_I^*)) = \bigsqcup_{j:\langle m, j\rangle_I = k} p_k(\mathcal{A}_{k,j}^{\pm}(f, E_I^*)),$$

where $\mathcal{A}_{k,j}^{\pm}(f, E_I^*) := \{\gamma \in \pi_*^{-1}(\mathcal{A}_k^{\pm}(f)) \cap \mathcal{L}(M, E_I^{*,\pm}) : \operatorname{ord}_\gamma E_i = j_i\}$. Let $p \in E_I^*$ and let U be a coordinate neighbourhood at p. Using the local coordinates $y = (y_1, \ldots, y_n) : U \to \mathbf{R}^n$ with $E_I^* = \{y_i = 0, i \in I, y_i \neq 0, i \notin I\}$, we can express $f\circ\pi$ as follows:

$$f\circ\pi(y) = u(y)\prod_{i\in I} y_i^{m_i}, \qquad \text{where } u(y) \text{ is a unit.}$$

We set $y_I = (y_i)_{i\in I}$ and define

$$\widehat{E}_I^{\pm}|_U = \Big\{(p, y_I) \in (E_I^* \cap U)\times \mathbf{R}^{|I|} : u(p)\prod_{i\in I} y_i^{m_i} = \pm 1\Big\}$$

The sets $\widehat{E}_I^{\pm}|_U$ can be patched together and we obtain a set $\widehat{E}_I^{\pm}$. We denote by m_I the greatest common divisor of m_i, $i \in I$, and define

$$\widetilde{E}_I^{\pm}|_U = \{(p, w) \in (E_I^* \cap U)\times\mathbf{R} : u(p)w^{m_I} = \pm 1\}.$$

The sets $\widetilde{E}_I^{\pm}|_U$ can be patched together and we obtain a set $\widetilde{E}_I^{\pm}$. Setting $\tilde{\beta}_I^{\pm} = \beta(\widetilde{E}_I^{\pm})$, we obtain

$$Z_f^{\pm}(t) = \sum_I \tilde{\beta}_I^{\pm}\,(u-1)^{|I|-1}\prod_{i\in I}\phi\Big(\frac{t^{m_i}}{u^{\nu_i}}\Big).$$

Theorem 5.12 ([27]). *Let $f, g : \mathbf{R}^n, 0 \to \mathbf{R}, 0$ be two analytic functions and let β be the $\mathbf{Z}/2\mathbf{Z}$-Euler characteristic with compact supports. Assume that there are real modifications $\pi_i : M_i \to \mathbf{R}^n$, $i = 1, 2$, so that π_1 (resp. π_2) is an isomorphism except possibly over the zero set of f (resp. g). If there is an analytic isomorphism $(M_1, \pi_1^{-1}(0)) \to (M_2, \pi_2^{-1}(0))$ which induces a blow-analytic homeomorphism $h : \mathbf{R}^n, 0 \to \mathbf{R}^n, 0$ with $f = g\circ h$, then $Z_f(t) = Z_g(t)$, $Z_f^{\pm}(t) = Z_g^{\pm}(t)$.*

Similarly we obtain the following

Theorem 5.13 ([10]). *Let $f, g : \mathbf{R}^n, 0 \to \mathbf{R}, 0$ be two polynomial functions and let β be the virtual Poincaré polynomial. Assume that there are algebraic modifications $\pi_i : M_i \to \mathbf{R}^n$, $i = 1, 2$, whose critical loci are normal crossings. We assume that π_1 (resp. π_2) is an isomorphism except over the zero set of f (resp. g). If there is an analytic isomorphism $(M_1, \pi_1^{-1}(0)) \to (M_2, \pi_2^{-1}(0))$ which induces a blow-analytic isomorphism $h : \mathbf{R}^n, 0 \to \mathbf{R}^n, 0$ with $f = g \circ h$, then $Z_f(t) = Z_g(t)$, $Z_f^{\pm}(t) = Z_g^{\pm}(t)$.*

See Definition 7.2 below for the notion of blow-analytic isomorphism.

6. Lipschitz maps

An interesting class of maps which are not differentiable is the class of Lipschitz maps. We start with some basics.

Let U be a convex open subset of $\mathbf{R}^n$. A map $f : U \to \mathbf{R}^p$ is said to be *Lipschitz* if there is a positive constant K so that

$$|f(x) - f(x')| \leq K|x - x'| \qquad \forall x, x' \in U.$$

Recall that Rademacher's theorem ([15, Theorem 4.1.1]), states that a function which is Lipschitz on an open subset of $\mathbf{R}^n$ is differentiable almost everywhere (in the sense of Lebesgue measure) on that set. This allows us to introduce the following definition.

Definition 6.1 (Generalized Jacobian). The generalized Jacobian $\partial f(0)$ of f at 0 is the convex hull of all matrices obtained as limits of sequences of the Jacobi matrices of f at x_i where $x_i \to 0$, $x_i \notin Z$. Here Z denotes the set of points at which f fails to be differentiable.

Theorem 6.2 ([5]). *Let $f : \mathbf{R}^n, 0 \to \mathbf{R}^n, 0$ be a Lipschitz map-germ. If $\partial f(0)$ does not contain singular matrices, then f has a Lipschitz inverse.*

In this section, we are interested in blow-analytic maps satisfying the Lipschitz condition.

Let U be a convex open subset of $\mathbf{R}^n$ and let $f : U \to \mathbf{R}$ be a continuous function with subanalytic graph. Then there is an nowhere dense closed subanalytic subset Z so that f is analytic on $U - Z$.

Lemma 6.3. *The function f is Lipschitz if and only if all partial derivatives of f are bounded on $U - Z$.*

Theorem 6.4 ([13]). *Let $f : \mathbf{R}^n, 0 \to \mathbf{R}^n, 0$ be an arc-analytic map with subanalytic graph. If f is bilipschitz, i.e., there are positive constants c_1, c_2 so that*

$$c_1|y - y'| \leq |f(y) - f(y')| \leq c_2|y - y'|,$$

then f^{-1} is arc-analytic.

Let $f : \mathbf{R}^n, 0 \to \mathbf{R}^n, 0$ be a homeomorphism which is blow-analytic and Lipschitz. The theorem asserts that the inverse f^{-1} is blow-analytic, if f^{-1} is Lipschitz.

Corollary 6.5. *Let $f : \mathbf{R}^n, 0 \to \mathbf{R}^n, 0$ be an arc-analytic map with semi-algebraic graph. If f is bilipschitz, then f^{-1} is blow-analytic.*

Theorem 6.6 ([13]). *Let* $F : \mathbf{R}^m \times \mathbf{R}^n, 0 \to \mathbf{R}^n$, $(x, y) \mapsto F(x, y)$, *be an arc-analytic map with subanalytic graph. If there are positive constants* c_1, c_2 *so that*

$$c_1|y - y'| \le |F(x,y) - F(x,y')| \le c_2|y - y'|, \tag{1}$$

then there is an arc-analytic and subanalytic map $\tau : \mathbf{R}^m, 0 \to \mathbf{R}^n, 0$ *such that*

$$\{F(x,y) = 0\} = \{y = \tau(x)\}. \tag{2}$$

Remark 6.7. Let $\alpha = (\alpha_1, \dots, \alpha_n) : \mathbf{R}, 0 \to \mathbf{R}^n, 0$ be an analytic map. Let $\mathrm{ord}(\alpha)$ denote $\min\{\mathrm{ord}(\alpha_1), \dots, \mathrm{ord}(\alpha_n)\}$. If an arc-analytic map $f : \mathbf{R}^n, 0 \to \mathbf{R}^n, 0$ is Lipschitz, then $\mathrm{ord}(f \circ \alpha) \ge \mathrm{ord}(\alpha)$. If the map $f : \mathbf{R}^n, 0 \to \mathbf{R}^n, 0$ is bilipschitz, then $\mathrm{ord}(f \circ \alpha) = \mathrm{ord}(\alpha)$. In particular, the image of a nonsingular curve by an arc-analytic bilipschitz map $\mathbf{R}^n, 0 \to \mathbf{R}^n, 0$ is a nonsingular curve.

Question 6.8. Does there exist a blow-analytic map (or an arc-analytic map with subanalytic graph) $f : \mathbf{R}^n, 0 \to \mathbf{R}^n, 0$ with the following properties?

- there is a positive constant c so that
$$c|y - y'| \le |f(y) - f(y')| \qquad \forall y, y' \in \mathbf{R}^n, 0;$$
- f is not Lipschitz.

7. Blow-analytic isomorphism and analytic arcs

A blow-analytic homeomorphism can be quite far from a bilipschitz homeomorphism.

Theorem 7.1 ([26]). *For any unibranched curve* $C \subset \mathbf{R}^2, 0$, *there is a blow-analytic homeomorphism* $h : \mathbf{R}^2, 0 \to \mathbf{R}^2, 0$ *such that* $h(C)$ *is nonsingular.*

Theorem 7.1 motivates us to strengthen the conditions imposed to the definition of blow-analytic homeomorphisms.

Definition 7.2. We say that a map $f : \mathbf{R}^n, 0 \to \mathbf{R}^n, 0$ is a *blow-analytic isomorphism* (bai for short) if there are two neighbourhoods U, U' of 0 in $\mathbf{R}^n$ so that the following conditions are satisfied.

- there are complex modifications $\pi : M \to U$, $\pi' : M' \to U'$, and an analytic isomorphism $F : (M, E) \to (M', E')$ of analytic spaces, where E and E' denote the critical loci of π and π' respectively.
- f is a homeomorphism and $\pi' \circ F = f \circ \pi$.

A blow-analytic isomorphism is clearly a blow-analytic homeomorphism. But the converse is not true. For example, the blow-analytic homeomorphism in Example 7.1 is not a bai. In fact, the critical locus of the composites of horizontal arrows are normal crossing, and we have a correspondence between their irreducible components, but they have different multiplicities.

Let $\pi : M \to \mathbf{R}^n$ be a complex modification whose critical locus is a normal crossing divisor. We consider an analytic vector ξ on M which is tangent to each irreducible component of the critical locus. By integrating ξ, we obtain an analytic isomorphism of M. If it induces a homeomorphism of $\mathbf{R}^n$ near 0, this is a blow-analytic isomorphism. Thus, in all triviality theorems stated before, we can replace bah by bai.

Definition 7.3. Let $\pi : M \to U$ be a composition of blowing-ups with nonsingular centers. A blow-analytic function $P : U \dashrightarrow \mathbf{R}$ is said to be a *blow-analytic unit* (bau for short) via π if $P \circ \pi$ extends to an analytic unit (i.e. an analytic function which is nowhere vanishing). P is said to be a *blow-analytic unit* (bau for short) if there is $\pi : M \to U$ such that P is a bau via π

Theorem 7.4. *If $f : \mathbf{R}^n, 0 \to \mathbf{R}^n, 0$ is a blow-analytic isomorphism, then the Jacobian determinant* $\det(df)$ *is a blow-analytic unit.*

Let $w_1, \ldots, w_n$ be real numbers. We consider the map

(3) $\qquad f : \mathbf{R}^n, 0 \to \mathbf{R}^n, 0, \quad x = (x_1, \ldots, x_n) \mapsto (x_1 P(x)^{w_1}, \ldots, x_n P(x)^{w_n}),$

where $P : \mathbf{R}^n, 0 \dashrightarrow \mathbf{R}$ is a bounded blow-analytic function.

Theorem 7.5. *Let P be a non-negative blow-analytic function via some toric modification $\pi : M \to \mathbf{R}^n$. If $P + \sum_{i=1}^n w_i x_i \frac{\partial P}{\partial x_i}$ is a blow-analytic unit via the modification π, and if P and $\sum_{i=1}^n w_i x_i \frac{\partial P}{\partial x_i}$ are continuously extendable on $\mathbf{R}^n - 0, 0$, then the map f defined by* (3) *is a blow-analytic isomorphism.*

Example 7.6. The map

$$f : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0), \quad (x, y) \mapsto (xP^3, yP^2), \quad P = \frac{x^4 + 2y^6}{x^4 + y^6},$$

is a blow-analytic isomorphism.

Consider the map

$$f : \mathbf{R}^n, 0 \to \mathbf{R}^n, 0, \quad x = (x_1, \ldots, x_n) \mapsto (x_1 + Q(x_2, \ldots, x_n), x_2, \ldots, x_n),$$

where $Q : \mathbf{R}^{n-1}, 0 \to \mathbf{R}$ is a blow-analytic function. Since the map $(x_1, \ldots, x_n) \mapsto (x_1 - Q(x_2, \ldots, x_n), x_2, \ldots, x_n)$ is the inverse of f, f is a homeomorphism.

Theorem 7.7. *If Q is blow-analytic, then f is a blow-analytic isomorphism.*

Example 7.8 ([38]). Consider a blow-analytic map $f : \mathbf{R}^3, 0 \to \mathbf{R}^3, 0$ defined by

$$(x, y, z) \mapsto \left(x, y, z + \frac{2x^5 y}{x^6 + y^4}\right).$$

This is a blow-analytic isomorphism by Theorem 7.7. Let $\alpha : \mathbf{R}, 0 \to \mathbf{R}^3, 0$ be the map defined by $t \to (t^2, t^3, 0)$. Observe that $f \circ \alpha(t) = (t^2, t^3, t)$. This means that the blow-analytic isomorphism f sends a singular curve, the image of α, to a regular curve.

We say that an analytic map $\alpha : \mathbf{R}, 0 \to \mathbf{R}^n, 0$ is *irreducible* if α cannot be written as $\alpha = \beta \circ \psi$, where $\beta : \mathbf{R}, 0 \to \mathbf{R}^n, 0$ and $\psi : \mathbf{R}, 0 \to \mathbf{R}, 0$, are analytic and $\psi'(0) = 0$.

Theorem 7.9. *Let $\alpha : \mathbf{R}, 0 \to \mathbf{R}^n, 0$, $n \geq 3$, be an irreducible analytic map. Then there is a blow-analytic isomorphism $f : \mathbf{R}^n, 0 \to \mathbf{R}^n, 0$ such that $f \circ \alpha$ is a regular map.*

8. Jacobian of blow-analytic map

Let $f : \mathbf{R}^n, 0 \to \mathbf{R}^n, 0$ be a blow-analytic map. It is interesting to investigate what we can conclude when we assume that $\det(df)$ is a blow-analytic unit. For example, is such a f a blow-analytic isomorphism?

Example 8.1. We identify $\mathbf{R}^2$ with $\mathbf{C}$ by the map $(x, y) \mapsto z = x + \sqrt{-1}y$. Let k be a positive integer. Consider the continuous blow-analytic map

$$f : \mathbf{C}, 0 \to \mathbf{C}, 0, \qquad z \mapsto z^{k+1}/\bar{z}^k = z^{2k+1}/|z|^{2k}.$$

Looking at the restriction to a small circle $|z| = \varepsilon$, the mapping degree of f is $2k+1$. In particular, f is not a homeomorphism. Since

$$\det(df) = \begin{vmatrix} (k+1)z^k/\bar{z}^k & -kz^{k+1}/\bar{z}^{k+1} \\ -k\bar{z}^{k+1}/z^{k+1} & (k+1)\bar{z}^k/z^k \end{vmatrix} = (k+1)^2 - k^2 = 2k+1, \quad z \neq 0,$$

$\det(df)$ is a blow-analytic unit. We also have that f is Lipschitz, by Lemma 6.3. Let $M \to \mathbf{C}$ denote the blowing-up at the origin. Since the map f is induced by an unbranched covering $M \to M$ of degree $2k+1$, f has the arc lifting property.

Example 8.1 shows that a blow-analytic map $f : \mathbf{R}^n, 0 \to \mathbf{R}^n, 0$ may not be a homeomorphism, even though $\det(df)$ is a blow-analytic unit. However, this kind of phenomenon is not possible in higher codimensional cases.

Proposition 8.2. *Let $f : \mathbf{R}^n, 0 \to \mathbf{R}^n, 0$ be a blow-analytic map so that $\det(df)$ is a blow-analytic unit. If there is a subset C of $\mathbf{R}^n, 0$, of codimension ≥ 3, so that $f|_{\mathbf{R}^n - C}$ is analytic, then f is a homeomorphism.*

It is an open question whther f is a bai or not.

We have a version of the inverse mapping theorem via toric modification, which is the following

Theorem 8.3. *Let $h = (h_1, \ldots, h_n) : \mathbf{R}^n, 0 \to \mathbf{R}^n, 0$ be a continuous blow-analytic map via a toric modification. If $\frac{\partial h_1}{\partial x_1}, \frac{\partial(h_1,h_2)}{\partial(x_1,x_2)}, \ldots, \frac{\partial(h_1,\ldots,h_n)}{\partial(x_1,\ldots,x_n)}$ are blow-analytic units and they are continuously extendable on $\mathbf{R}^n - 0, 0$, then h is a blow-analytic isomorphism.*

If the map $h = (h_1, \ldots, h_n) : \mathbf{R}^n, 0 \to \mathbf{R}^n, 0$ satisfies the assumption of Theorem 8.3 after permutations of $x_1, \ldots, x_n$ and $h_1, \ldots, h_n$, then h is a blow-analytic isomorphism, by Theorem 8.3.

This is the corrected version of Theorem 6.1 in [12].

Lastly we have three more theorems.

Theorem 8.4. *Let $f : \mathbf{R}^n, 0 \dashrightarrow \mathbf{R}^n, 0$ be a blow-analytic map so that $\det(df)$ is a blow-analytic unit. If there are nonsingular subanalytic subsets C, C' so that f is blow-analytic via the blowing up with center C and that $f(C) = C'$, then $\operatorname{codim} C = \operatorname{codim} C'$ and f has the arc lifting property. Moreover, there is an analytic map $\tilde{f} : M \to M'$ such that $\tilde{f}$ is locally an isomorphism and that $\pi' \circ \tilde{f} = f \circ \pi$, where $\pi : M \to \mathbf{R}^n$ is the blowing-up at C and $\pi' : M' \to \mathbf{R}^n$ is the blowing-up at C'.*

Theorem 8.5. *Let $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ be a blow-analytic map. If $\det(df)$ is a blow-analytic unit, then f is finite.*

Theorem 8.6. *Consider a blow-analytic map* $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ *defined by*

$$(x_1, \ldots, x_n) \mapsto (x_1 P_1(x), \ldots, x_n P_n(x)), \quad \text{where } P_i \text{ are blow-analytic units.}$$

If f *is blow-analytic via a toric modification and* $\det(df)$ *is a blow-analytic unit, then* f *is a blow-analytic isomorphism.*

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